Eigenvalues of Symmetric Elliptic Operators

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Contents

A word of warning: The C^1 that I use here is really just Gateaux differentiable

1 Quantum Mechanics

I would be remiss if I did not mention one of the most important aspects of eigenvalues and eigenfunctions in mathematics. The entire field of functional analysis was started in part to deal with the abstract properties of quantum mechnical systems, since it turns out that infinite dimensional Hilbert spaces turn out to be the natural setting for particles with infinite possible states.

The approach I will take is that of Schrödinger, where the particles evolve in time, rather than that of Heisenberg where the observables evolve in time.

Now, the idea starts like this: imagine you have a particle confined to a bounded region in $\mathbb{R}$, where the potential energy of outside this region is infinite and inside it is 0. We will assume the bounded region is the interval $I = (-L/2, L/2)$. Then the particle evolves according the Schrödinger equation

$$\begin{cases} u_t = \lambda i u_{xx} & (x,t) \in I \times \{t > 0\} \\ u = 0 & (x,t) = \pm L/2 \times \{t \geq 0\} \\ u = u_0 & (x,t) \in I \times \{t = 0\} \end{cases}$$

where the boundary conditions come from the fact that a particle with finite energy cannot move to a location with infinite potential energy, and $0 \neq \lambda \in \mathbb{R}$. On the other hand suppose that $v = v(x)$ sovles

$$\begin{cases} \lambda v_{xx} = -k^2 v & x \in I \\ u = 0 & x = \pm L/2 \end{cases}$$

then $u(x,t) = v(x)e^{-it\omega}$ has $u_t = -i\omega u$ and $\lambda u_{xx} = -k^2 u$ hence

$$u_t - \lambda i u_{xx} = -i(\omega u - k^2 u) = 0 \Leftrightarrow \omega = k^2$$

$$u|_{\pm L/2 \times \{t \geq 0\}} = 0$$

$$u|_{t=0} = v$$

and so we see that the solution to the time evolving Schrödinger equation is highly related to the eigenvalue problem for the related steady-state equation.

In a general quantum system we have a Hilbert space X and a hamiltonian $H : D(H) \subseteq X \to X$ which is a linear operator such that the domain $D(H) = \{x \in X : \|Hx\|_X < \infty\}$ is a dense subspace of X. The space H is the "particle" while elements in $D(H)$ of magnitude 1 are called states, which are essentially the allowable configurations the particle can be in. We also identify the state x with the state $-x$, or in the complex case with $e^{i\theta}x, \forall \theta \in \mathbb{R}$.

We may make a few more assumptions that I won't clarify, as they will come up later, so that H is symmetric($\langle Hx, y \rangle_X = \langle x, Hy \rangle_X$ for $x, y \in D(H)$) and there exists a countable orthonormal basis $\{e_i\}_{i=1}^{\infty}$ such that $He_i = \lambda_i e_i$ where $\lambda_i \geq 0$ and $\lambda_i \leq \lambda_{i+1}$.

The fundamental postulate of the Schrödinger picture of quantum mechanics is that for each state $x = x(t)$, the time evolution of the particle stating in that state is given by

$$i\hbar x_t = Hx \tag{1}$$

This is a Banach space-valued ordinary differential equation, and so we have some analogies with finite-dimensional vector ODEs that will be explored in the next section.

Now, often the Hilbert space under consideration is relatively concrete, rather than abstract so that calculations are easier, and sometimes don't need a Hamiltonian in extremely simple cases. An example is that of a particle being shot through a device that polarizes it so that the particle hits one of two spots on a luminescent screen.

Then the two basis states would be the two spots, while other states give a linear combination of the two. Hence we have that the Hilbert space is exactly a two dimensional vector space.

Another way to construct such a Hilbert space is through the process of *quantization*, whereby the quantum space is obtained from a classical configuration space.

Consider a particle moving in an open bounded domain U in $\mathbb{R}^n$, with spatial position being $q = (q_1, ..., q_n)$ and corresponding momentum $p = (p_1, ..., p_n)$. We have that $p_i = mv_i = m\dot{q}_i$ where $\dot{q}_i = \frac{dq_i}{dt}$ and $H = T + V$, where $V \geq 0$ is the potential energy and T is the kinetic energy. Once again the bounded region can be though of as being where the potential energy is finite, or in other words $U = \{x : V(x) < \infty\}$ and we see that $V(x) = \infty$ on the boundary by this description.

We may rewrite this using the fact that $T = \frac{1}{2}mv^2 = \frac{p \cdot p}{2m} = \frac{1}{2m}\sum_i p_i^2$, from which quantization yields the operators $q_i \mapsto \hat{q}_i = x_i$, as multiplication, and $p_j \mapsto \hat{p}_j = -i\hbar\partial_{x_j}$, to obtain that the corresponding quantum Hamiltonian operator is $\hat{H} = -\frac{\hbar^2}{2m}\triangle + V(x)$, which we immediately see is an elliptic partial differential operator.

Along with this the Schrödinger equation (1) yields $i\hbar x_t = \hat{H}x$ with the Hilbert space $X = L^2(U)$ and natural domain

$$D(H) = H^2(U) \cap H_0^1(U)$$

(to incorporate the boundary conditions that the particle has 0 probability of being in locations with infinite potential energy), if we assume that the coefficients and the boundary are smooth. But this is simply the differential equation

$$\begin{cases} i\hbar u_t = -\frac{\hbar}{2m}\triangle u + V(x)u & (x,t) \in U \times \{t > 0\} \\ u = 0 & (x,t) \in \partial U \times \{t \geq 0\} \\ u = u_0 & (x,t) \in U \times \{t = 0\} \end{cases} \tag{2}$$

and our earlier discussion leads us to study the related eigenvalue problem

$$\begin{cases} -\frac{\hbar}{2m}\triangle u + V(x)u = \lambda u & x \in U \\ u = 0 & x \in \partial U \end{cases} \tag{3}$$

2 Functional Calculus

The ideas connecting the partial differential equation (2) and the eigenvalue problem (3) come at no great surprise. Consider an n-dimensional vector space, let's say $\mathbb{R}^n$(or $\mathbb{C}^n$, the analysis carries over) endowed with the usual inner product and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

We recall that if A is symmetric(that is $Ax \cdot y = x \cdot Ay$) then the spectral theorem holds: there exists an orthonormal basis of eigenvectors and they have corresponding eigenvalues, $\{(e_i, \lambda_i)\}_{i=1}^n$.

On the other hand consider the matrix ordinary differential equation $\dot{X} = AX$. Then if we define

$$e^{tA}X = \sum_{i=0}^{\infty} \frac{t^n A^n}{n!} X$$

it holds that the operator norm

$$\left\| \frac{t^n A^n}{n!} \right\| \leq \frac{t^n \|A\|^n}{n!} \tag{4}$$

so

$$\left\| e^{tA} \right\| \leq e^{t\|A\|} < \infty \Rightarrow \left\| e^{tA} X \right\| \leq e^{t\|A\|} \|X\|$$

and so by completeness of the space of linear maps we see that e^{tA} is a bounded linear map and converges uniformly. Furthermore, we have that

$$\frac{d}{dt} e^{tA} X = \sum_{i=0}^{\infty} \frac{t^{n-1} A^n}{(n-1)!} X = A(e^{tA} X)$$

so that we have

Theorem. *The unique solution to the matrix ordinary differential equation $\dot{X} = AX$, with A a general matrix, with initial condition X_0 is $e^{tA} X_0$.*

A reasonable question is how we generalize this to the infinite-dimensional setting. Inequality (4) clues us in to the fact that if $A : Y \to Y$ is a bounded operator on a Banach space, then once again

$$\left\| \frac{t^n A^n}{n!} \right\| \leq \frac{t^n \|A\|^n}{n!}$$

and hence

$$e^{tA}X = \sum_{i=0}^{\infty} \frac{t^n A^n}{n!} X$$

is well-defined. Furthermore, we see immediately that the solution to the Banach space ODE with initial X_0 is $e^{tA} X_0$ by a similar argument to before. Thus we have in direct analogy with the above theorem:

Theorem. *The unique solution to the Banach space ordinary differential equation $\dot{X} = AX$, with A a bounded operator, with initial condition X_0 is $e^{tA}X_0$.*

This then begs the question, **what do we do with unbounded operators**?

To answer this we suppose that A is symmetric on $\mathbb{R}^n$ and has an orthonormal basis of eigenvectors and with corresponding real eigenvalues, $\{(e_i, \lambda_i)\}_{i=1}^n$. There exists an invertible matrix B(a change-of-basis one) such that $B(x) = \sum_{i=1}^n \langle x, e_i \rangle e_i$ and

$$B^{-1}AB = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = C$$

Note that we then have $(B^{-1}AB)^n = B^{-1}A^nB$ and so $e^{t(B^{-1}AB)} = B^{-1}e^{tA}B$. On the other hand we have that $e^{tC} = \begin{bmatrix} e^{t\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{t\lambda_n} \end{bmatrix}$ and hence

$$e^{tA} = \sum_{i=1}^{n} \langle x, e_i \rangle e^{t\lambda_i} e_i$$

Of course in the infinite dimensional case $|e^{t\lambda_i}| \leq C < \infty$ for all i, so that this series may converge.

This leads to the following theorem from, say [4] proposition 5.30(or from MAT436 or possibly MAT437):

Theorem 1. *Suppose that $L : D(H) \subseteq X \to X$ is a densly defined symmetric operator on a Hilbert space. Suppose further that there exists an orthonormal basis of X consisting of eigenvectors and eigenvalues $\{e_i, \lambda_i\}$ such that the eigenvalues are bounded above. Then a solution to the Banach space ordinary differential equation $\dot{X} = LX$ is given by $X(t) = e^{tA}X_0 = \sum_{i=1}^{\infty} \langle X_0, e_i \rangle e^{t\lambda_i} e_i$*

A similar analysis holds for $i\dot{X} = LX$ and $e^{it\lambda_i}$.

This shows the fundamental relationship between the heat(where the operator is $e^{-t\triangle}$) and Schrödinger(where the operator is $e^{it\triangle}$) equations , and the eigenvalues for the corresponding elliptic problem.

Furthermore, this makes the semigroup properties of $e^{-t\triangle}$(and $e^{it\triangle}$) obvious. This is because

$$e^{-t\triangle}e^{-s\triangle}X_0 = \sum_{i=1}^{\infty} \langle X_0, e_i\rangle e^{-t\lambda_i}e^{-s\lambda_i}e_i = \sum_{i=1}^{\infty} \langle X_0, e_i\rangle e^{-(t+s)\lambda_i}e_i = e^{-(t+s)\triangle}X_0$$

as the e_i are pairwise orthogonal, and

$$e^{-0\triangle}X_0 = \sum_{i=1}^{\infty} \langle X_0, e_i\rangle e^{-0\lambda_i}e_i = \sum_{i=1}^{\infty} \langle X_0, e_i\rangle e_i = X_0$$

3 Weak Convergence and Constraints

Now, we consider a model problem to motivate the more general statement.

Suppose we have a domain $D \subseteq \mathbb{R}^n$ and a map $f : \overline{D} \to \mathbb{R}$. We want to know whether $\min_{x\in D} f(x)$ exists, is finite, and if there is some $y \in \overline{D}$ such that $f(y) = \min_{x\in D} f(x)$.

The first part is obvious, there is such a minimum, though it is possible that it is equal to $-\infty$.

Example 2. For example, take $f(x, y) = -|x|^2$ with domain $D = \mathbb{R}^n$.

With regards to the finite aspects we have two approaches. One assumes that D is finite and f is continuous(otherwise enumerate the rationals $\{q_n\}$ in $[0, 1]$ and consider $f(q_n) = -n, f(x) = 0$ otherwise). To see this is sufficient we note that f is a continuous function on a compact set($\overline{D}$) and thus achieves its finite minimum on that set.

Another, much weaker, sufficient condition is that $\lim_{|x|\to\infty} f(x) = \infty$, or to put it another way, for each $M > 0$ there exists $r > 0$ such that for each $x \in D \cap \{|x| > r\}$ $f(x) > M$. If D is bounded then this is vacuously true.

Example 3. By picking $M = 1$ for example

$$\min_{x\in D} f(x) \geq \min\{M, \min_{x\in\bar{D}\cap\{|x|\leq r\}} f(x)\}$$

and since the set $x \in \bar{D} \cap \{|x| \leq r\}$ is compact, we once more have that by continuity $\min_{x\in D} f(x) > -\infty$. In fact, the minimum is acheived, since $\infty > \min_{x \in D} f(x) > -\infty$ and so picking $M > \min_{x \in D} f(x) + 1$ it holds that

$$\min_{x \in D} f(x) = \min\{M, \min_{x \in \bar{D} \cap \{|x| \leq r\}} f(x)\} = \min_{x \in \bar{D} \cap \{|x| \leq r\}} f(x)$$

which by continuity again has the desired property.

However, if f is merely lower-semicontinuous and has the property that $\lim_{|x| \to \infty} f(x) = \infty$ then the minimum exists and is acheived. To see this consider a minimizing sequence $\{x_n\} \subseteq D$. Since $\lim_{|x| \to \infty} f(x) = \infty$ it holds that the sequence must lie in a bounded set. Thus by compactness of bounded sets there is a convergent subsequence $x_{n_k} \to x$. By lower-semicontinuity then it holds that

$$f(x) \leq \liminf_k f(x_{n_k}) \leq \liminf_n f(x_n) = \min_{y \in D} f(y)$$

On the other hand since $\bar{D}$ is closed $x \in \bar{D}$ and hence $f(x) > -\infty$ as it is a real number.

In summary, we have:

Theorem 4. *Suppose that we have a domain $D \subseteq \mathbb{R}^n$ and a lower-semicontinuous function $f : \overline{D} \to \mathbb{R}$ such that $\lim_{|x| \to \infty} f(x) = \infty$ in the manner as described above. Then $\min_{y \in D} f(y)$ exists and is finite, and there exists $x \in \overline{D}$ such that $f(x) = \min_{y \in D} f(y)$*

Now, an immediate generalization of this to Hilbert spaces goes as follows:

Claim. If $D \subseteq X$ is a domain, X a Hilbert space, and $f : \bar{D} \to \mathbb{R}$ is a lower-semicontinuous function, and furthermore $\lim_{|x| \to \infty} f(x) = \infty$. Then $\min_{y \in D} f(y)$ exists and is finite, and there exists $x \in \overline{D}$ such that $f(x) = \min_{y \in D} f(y)$

However, this is false.

Example 5. Let $X = c_0$ with $D = \{e_i\}_{i=1}^{\infty}$ and $f(e_i) = \frac{1}{i}$, where e_i is the sequence with 1 in the i-th spot and 0's elsewhere. This is a discrete, bounded, and closed, set so in particular f is continuous(and so lower-semicontinuous). Then $\min_i f(e_i) = 0$ but as this set has no limit points, a simple exercise, it follows that the minimum is not acheived.

This failure is a result of the lack of compact sets in infinite dimensional Hilbert spaces, specifically that closed and bounded sets are no longer compact. However, since Hilbert spaces are reflexive, in the weak topology bounded sets are closed. The problem in this case is that $\bar{D}$ may not be weakly closed(in example 5 $e_i \rightharpoonup 0$ weakly), and f may not be lower-semicontinuous in the weak topology(i.e. weakly lower-semicontinuous).

However, we immediately have

Theorem 6. *Suppose that f is weakly lower-semicontinuous, $D \subseteq X$ is weakly closed and $\lim_{|x|\to\infty} f(x) = \infty$. Then $\min_{y\in D} f(y)$ exists and is finite, and there exists $x \in D$ such that $f(x) = \min_{y\in D} f(y)$*

Proof. As before $\min_{y\in D} f(y)$ trivially exists, and let $\{x_n\} \subseteq D$ be a minimizing sequence. Since $\lim_{|x|\to\infty} f(x) = \infty$ there exists some $r > 0$ so that $|x| > r$ implies $f(x) > \min_{y\in D} f(y) + 1$. Thus for large enough n $x_n \in B_r(0)$, so upon relabling we may assume $\{x_n\} \subseteq B_r(0)$. Then there must exist a weakly convergent subsequence $x_{n_k} \rightharpoonup x$, and since $D \subseteq X$ is weakly closed it holds that $x \in D$.

By weak lower-semicontinuity we have that

$$f(x) \leq \liminf_k f(x_{n_k}) \leq \liminf_n f(x_n) = \min_{y\in D} f(y)$$

but since $x \in D$ we must have that $-\infty < f(x) = \min_{y\in D} f(y)$ □

Now, we want to apply this to differential equations. We first start by finding an example of a weakly closed subset, following [1] chapter 8.

Lemma 7. *Suppose that $G \in C^\infty(\mathbb{R})$ is such that $|G(x)| \leq C(|x|^2 + 1)$ and $|G'(x)| \leq C(|x| + 1)$ for some $C > 0$. Then the set $\{u \in H_0^1(U) : \int G(u) = 1\}$ where $U \subseteq \mathbb{R}^n$ is bounded, is weakly closed.*

Proof. Suppose that $u_n \rightharpoonup u$ weakly in $H_0^1(U)$, which by standard functional analysis results implies $\{u_n\}$ is bounded in $H_0^1(U)$(use the uniform boundedness principle on $v \mapsto \langle u_n, v\rangle$). By the Sobolev embedding theorem this contains a sequence converging strongly in L^2, and by uniqueness of weak limits this strong limit is also u. Now, we have that(if $u_n \geq u$, the other case is similar)

$$|G(u_n) - G(u)| \leq \int_u^{u_n} |G'(x)|dx \leq \int_u^{u_n} C(|x| + 1)dx$$

$$\leq C(u_n - u)(1 + |u| + |u_n|)$$

and hence $|G(u_n) - G(u)| \leq C|u_n - u|(1 + |u| + |u_n|)$

Now, we have that upon relabling the sequence so $u_n \to u$ strongly in L^2:

$$|\int G(u_n) - \int G(u)| \leq \int |G(u_n) - G(u)| \leq Cvol(U) \left\|u_n - u\right\|_2 \left\|1 + |u| + |u_n|\right\|_2$$

which goes to 0. $\square$

Now, let us give a reason why this is important. We have that through some elementary calculations, that are more here for heuristics:

Claim. Let $I \in D \subseteq C^1(H_0^1)$(for all $u \in D$ and $v \in H_0^1$ $\frac{d}{dt}|_{t=0} I(u+tv) = D_u I(v)$ exists and is finite) satisfy the requirements of theorem 6, where $D = \{u \in H_0^1(U) : \int G(u) = 1\}$ for G as in lemma 7. Then there exists a real number λ such that $D_u I(v) = \lambda \int G'(u)v$ where u is the minimizer of $\min_{y \in D} f(y)$ i.e. $f(u) = \min_{y \in D} f(y)$

Proof. We have that for each $v \in H_0^1$ the function $t \mapsto I(u+tv)$ exists and is differentiable at 0. Since u minimizes it holds that this function has a minimum at 0, hence

$$0 = \frac{d}{dt}|_{t=0} I(u+tv) = \frac{d}{dt}|_{t=0} \frac{I(u+tv)}{\int G(u+tv)} = \frac{D_u I(v) \int G(u) - I(u) \int G(u)v}{(\int G(u))^2}$$

so

$$0 = D_u I(v) \int G(u) - \min_{y \in D} f(y) \int G(u)v$$

$\square$

However, this proof doesn't hold up as $u + tv$ may not be in D and so the critical point argument isn't formal. However, this will provide inspiration for later calculations. A formal proof can be found in [1] chapter 8.

4 Eigenvalues

Let U be a bounded domain in $\mathbb{R}^n$ with smooth boundary and $H_0^1 = H_0^1(U)$.

Recall the two energy estimates for uniformly elliptic operators of the form $Lu = -(a_{ij}u_i)_j + cu$ with $c \geq 0$ and all coefficients smooth up to the boundary, and with B the associated bilinear form on H_0^1:

1. There exists a constant $C > 0$ such that $|B[u, v]| \leq C \|u\|_{H_0^1} \|v\|_{H_0^1}$
2. There exists another constant $K > 0$ such that $B[u, u] \geq K \|u\|_{H_0^1}^2$

It also holds in this case that L is a symmetric operator on H^2 and B is a symmetric bilinear form.

Observe for later use the following fact:

Fact 8. *The bilinear form B is an inner product on H_0^1 inducing the same topology*

On the other hand we have the following useful theorem which I will not prove and which is a minor variation of the one in [1]:

Theorem 9. *Let $S : L^2 \to L^2$ be linear, symmetric, injective and compact. Suppose further that it is positive, or $\langle Sx, x\rangle_{H_0^1} \geq 0$. Then there exists a countable orthonormal basis of L^2 consisting of eigenvectors and eigenvalues $\{e_i, \lambda_i\}$ for S such that $\lambda_i > 0$, $\lambda_i \geq \lambda_{i+1}$ and they accumulate only at 0.*

On the other hand Lax-Milgram implies that for each $f \in L^2$ there exists a function $u \in H_0^1$ such that u is the unique weak solution of

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

We also know that u is characterized by $B[u, v] = \langle f, v\rangle_{L^2}$ for all $v \in H_0^1$. We then put $Sf = u$ in this case. We note that for all $v \in H_0^1$ it holds that

$$B[S(af + g), v] = \langle af + g, v\rangle_{L^2} = a\langle f, v\rangle_{L^2} + \langle g, v\rangle_{L^2}$$

so that S is linear. On the other hand we have that

$$C\|Sf\|_{H_0^1}^2 \leq B[Sf, Sf] = \langle f, Sf\rangle_{L^2} \leq \|f\|_2 \|Sf\|_2 \leq \|f\|_2 \|Sf\|_{H_0^1}$$

so S is bounded.

We actually have more than this, from [1] chapter 6 section 3 theorem 4:

Proposition 10. *Under all of the asserted conditions above, it holds that $S : L^2 \to H^2$ and it is bounded. Furthermore, it holds that $S = L^{-1}$ where $L : H^2 \to L^2$.*

Proof. The theorem in Evans implies $S : L^2 \to H^2$. On the other hand it states that $\|Sf\|_{H^2} \leq C(\|f\|_2 + \|Sf\|_2)$, but $\|Sf\|_2 \leq \|Sf\|_{H_0^1} \leq C\|f\|_2$ and hence $\|Sf\|_{H^2} \leq C\|f\|_2$.
The last part follows immediately from the fact that by integration by parts for $f \in L^2, u \in H^2$ it holds that for all $v \in H_0^1$ $\langle LSf, v\rangle_2 = B[Sf, v] = \langle f, v\rangle_2$ and $B[u, v] = \langle Lu, v\rangle_2 = B[SLu, v]$. Hence as both $\langle f, v\rangle_2$ and $B[u, v]$ are inner products on their respective space, it holds that $f = LSf$ and $u = SLu$ as required. □

Lemma 11. *Suppose that $T : X \to Y$ is bounded between Banach spaces and $S : Y \to Z$ is compact between Banach spaces. Then $ST : X \to Z$ is compact.*

Proof. Let $B \subseteq X$ be bounded. Then $T(B)$ is also bounded and so $S(T(B))$ is precompact. On the other hand $S(T(B)) = (ST)(B)$ and so the image of a bounded set is precompact. □

But since $H^2 \hookrightarrow L^2$ is compact, it holds that $S : L^2 \to L^2$ is compact.
Now, since it holds that S is linear, injective and compact to apply theorem 9 we simply need to show that is it symmetric and positive. Thus

Theorem 12. *Let L, S be as above. Then there exists a countable orthonormal basis of L^2 consisting of eigenvectors and eigenvalues $\{e_i, \lambda_i\}$ for S such that $\lambda_i > 0$, $\lambda_i \geq \lambda_{i+1}$ and they accumulate only at 0.*

Proof. To see that it is symmetric we let $u, v \in L^2$. Then

$$\langle Su, v\rangle_2 = \langle Su, LSv\rangle_2 = \langle LSu, Sv\rangle_2 = \langle u, Sv\rangle_2$$

as L is symmetric. Similarly

$$\langle Su, u\rangle_2 = \langle Su, LSu\rangle_2 = B[Su, Su] \geq 0$$

Thus by theorem 9 we are done □

Immediately from [1] chapter 6 section 3 theorem 5 we deduce the following chain for any i:

$$e_i \in L^2 = H^0 \Rightarrow Se_i \in H^2 \Rightarrow Su = \frac{1}{\lambda_i} S^2 u \in H^4 \Rightarrow ... \Rightarrow Su = \frac{1}{\lambda_i^n} S^{n+1} u \in H^{2(n+1)}$$

and so $u = \frac{1}{\lambda_i} Su \in \bigcap_{i=1}^{\infty} H^n = C^\infty$. Furthermore, since $S = L^{-1}$ where we view $L : H^2 \to L^2$ it follows that we have

Theorem 13. *Let L be as above. Then there exists a countable orthonormal basis of L^2 consisting of eigenvectors that are smooth, and eigenvalues $\{e_i, \lambda_i\}$ for L such that $\lambda_i > 0$, $\lambda_i \leq \lambda_{i+1}$ and they accumulate only at infinity.*

Proof. From theorem 12 we have a basis $\{f_i, \alpha_i\}$ of L^2 for S. But since $H^2 \subseteq L^2$ it should hold that $e_i = f_i \in H^2$ is such a basis. We check that $Lf_i = \frac{1}{\alpha_i} LSf_i = \frac{1}{\alpha_i} f_i$ so that these are eigenvectors with eigenvalues $\lambda_i = \frac{1}{e_i}$. To see that this is an orthonormal basis note that this follows by theorem 12. The other properties follow easily. $\square$

We could approach this using B and weak solutions by noting that $\alpha_i B[f_i, v] = B[Sf_i, v] = \langle f_i, v \rangle_2$ and then applying regularity theory from there, which is what Evans does.

Let us consider another situation. We look at the bilinear form B on the space $\dot{H} = \{u \in H^1 : \int_U u = 0\}$, the space of Sobolev functions with mean 0. We have that

Lemma 14. *The Bilinear form B is continuous and coercive over $\dot{H}$*

Proof. The proof of continuity is much the same as for the case of H_0^1. On the other hand we now need the Poincare inequality from [1] chapter 5 section 8 theorem 1. From these the conclusion follows. $\square$

Lemma 15. *The space $H^1 = \{u + c : u \in \dot{H}$ and $c = const\} = \dot{H} \oplus \mathbb{R}$*

Proof. Let $u \in H^1$, then $\int (u - \frac{1}{vol(U)} \int u) = \int u - \int u = 0$ and $\frac{1}{vol(U)} \int u \in \mathbb{R}$ $\square$

In this case the analysis carries over to the complex case too.

We have that from the reference [2] theorem 7.32 the exact same regularity as for the Dirichlet case above. Hence following through the analysis there is an orthonormal basis for L^2 consisting of eigenvalues and eigenvectors, with the same properties as before, for the operator

$$\begin{cases} Lu = f & \text{in } U \\ \partial_n f = 0 & \text{on } \partial U \end{cases}$$

in the Neumann problem, where ∂_n is the normal derivative. Furthermore, it holds that by that same theorem each eigenvector $e_i \in C^\infty(\bar{U})$ and $\partial_n f = 0$.

In summary,

Theorem 16. *Let $L : H^2 \to L^2$ be the natural extension of the operator*

$$\begin{cases} Lu = f & \text{in } U \\ \partial_n f = 0 & \text{on } \partial U \end{cases}$$

on smooth functions.

Then there exists a countable orthonormal basis of L^2 consisting of eigenvectors smooth up to the boundary, and eigenvalues $\{e_i, \lambda_i\}$ for L such that $\lambda_i > 0$, $\lambda_i \leq \lambda_{i+1}$ and they accumulate only at infinity.

5 A Brief Return to Quantum Mechanics

Recall that the hamiltonian H on a particle moving in a bounded domain $U \subseteq \mathbb{R}^n$ with smooth boundary often yields problems of the form

$$\begin{cases} Hu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

where $Hu = -\frac{\hbar}{2m}\triangle u + V(x)u$ and $V \geq 0$.

This justifies the $V(x)$ term being called the potential, since it is simply the potential energy in a quantum mechanical standpoint. Now, the pure energy states that a particle can acheive are the eigenvalues of the hamiltonian problem, and hence by the prior section we can say a lot about the energy.

First we have that the possible pure energy states is discrete. Along with this they are bounded below by 0, and go to infinity.

Thus we may apply the results of the second section to conclude that given a particle X_0 the solution to the time evolution for the Schrödinger equation is $X(t) = \sum_{i=1}^{\infty} e^{it\lambda_i} \langle X_0, e_i \rangle_2 e_i$. This immediately yields that $\|X(t)\|_2^2 = \sum_{i=1}^{\infty} |e^{-it\lambda_i} \langle X_0, e_i \rangle_2|^2$ by Plancherel's theorem. But $|e^{-it\lambda_i} \langle X_0, e_i \rangle_2|^2 = |\langle X_0, e_i \rangle_2|^2$ and hence it follows that

Theorem. *The total probability of X_0(or in other words, the L^2 norm) is conserved by the Schrödinger equation.*

Proof. It holds that

$$\|X(t)\|_2^2 = \sum_{i=1}^{\infty} |e^{-it\lambda_i} \langle X_0, e_i \rangle_2 |^2 = \sum_{i=1}^{\infty} | \langle X_0, e_i \rangle_2 |^2 = \|X_0\|_2^2$$

□

Another result that was in an early problem set is the following: given heat distribution X_0 the solution to the time evolution for the heat equation $u_t - \triangle u = 0$ is $X(t) = \sum_{i=1}^{\infty} e^{-t\lambda_i} \langle X_0, e_i \rangle_2 e_i$. Thus it follows that:

Theorem. *The L^2 norm of a solution to the heat equation decreases at least as fast as $e^{-t\lambda_1}$ where λ_1 is the first eigenvalue of the elliptic operator*

Proof. We have that

$$\|X(t)\|_2^2 = \sum_{i=1}^{\infty} |e^{-t\lambda_i} \langle X_0, e_i \rangle_2 |^2 \leq e^{-2t\lambda_1} \sum_{i=1}^{\infty} | \langle X_0, e_i \rangle_2 |^2 = e^{-2t\lambda_1} \|X_0\|_2^2$$

□

6 Rayleigh Quotients

Now that all of the abstract stuff is out of the way we can apply our results from section 3 to the problem of Eigenvalues. Consider the problem $\inf_{u \in H} B[u, u]$, where the space $H = \{u \in H_0^1 : \|u\|_2 = 1\}$. Then we have that from [1] chapter 6 section 5 theorem 2:

Theorem 17. *It holds that $\lambda_1 = \inf_{u \in H} B[u, u]$*

Proof. |

($\geq$) Consider e_1. Then $e_1 \in H_0^1$ and $\|e_1\|_2 = 1$ by construction, so $e_1 \in H$. On the other hand $B[e_1, e_1] = \lambda_1 \langle e_1, e_1 \rangle_2 = \lambda_1$ by construction

($\leq$) Consider that by definition of an orthonormal basis $f \in L^2$ can be written $f = \sum_{i=1}^{\infty} \langle f, e_i \rangle_2 e_i$ and so this holds true in particular for $f \in H$. On the other hand we know that B is an inner product on H_0^1, so we try to find an orthonormal basis.

Let $w_i = \frac{e_i}{\sqrt{\lambda_i}}$, then $B[w_i, w_j] = \frac{\lambda_i}{\sqrt{\lambda_i \lambda_j}} \langle e_i, e_j \rangle_2 = \delta_i^j$ so this is orthonormal. We also have that $B[w_i, g] = 0$ iff $\sqrt{\lambda_i} \langle e_i, g \rangle_2 = 0$ iff $\langle e_i, g \rangle_2 = 0$, since $\lambda_i > 0$ for all i. But by completeness this results in the statement $B[w_i, g] = 0$ for all i iff $\langle e_i, g \rangle_2 = 0$ for all i iff $g = 0$ so this is an orthonormal basis. We then have that $f = \sum_{i=1}^{\infty} \langle f, e_i \rangle_2 e_i = \sum_{i=1}^{\infty} \lambda_i \langle f, w_i \rangle_2 w_i$. But $\frac{1}{\sqrt{\lambda_i}} \langle f, e_i \rangle_2 = \langle f, w_i \rangle_2 = \frac{1}{\sqrt{\lambda_i}} B[w_i, f]$, so that $f = \sum_{i=1}^{\infty} \sqrt{\lambda_i} B[w_i, f] w_i$ is the unique decomposition.

Thus in conclusion by Plancherel's theorem

$$B[f, f] = \sum_{i=1}^{\infty} \lambda_i \langle f, e_i \rangle_2 \langle f, e_j \rangle_2 B[w_i, w_j] = \sum_{i=1}^{\infty} \lambda_i |\langle f, e_i \rangle_2|^2 \geq \lambda_1 \sum_{i=1}^{\infty} |\langle f, e_i \rangle_2|^2$$

$$= \lambda_1 \|f\|_2^2 = \lambda_1$$

□

However, we have something very nice here. Let's forget that e_1 acheives this minimum. We want to know then without abstract principles if $\inf_{u \in H} B[u, u]$ is acheived, and the properties of minimizers.

Let G be the smooth function $G(x) = |x|^2 = x^2$, with $|G'(x)| = 2|x|$. Then it satisfies the conditions of lemma 7, so that H is weakly closed. On the other hand let $Q(u) = B[u, u]$ the induced quadratic form. We have that:

Lemma 18. *Let a quadratic form $q(u)$ be induced by a bilinear form $p[u, v]$ on a Banach space X. Suppose that $q(u) = p[u, u] \geq C \|u\|_X^2$ and $p[u, v] \leq K \|u\|_X \|v\|_X$ for some $C, K > 0$. Then q is weakly lower-semicontinuous.*

Proof. We have that if $u_n \rightharpoonup u$ weakly in X then for all $v \in X$ we have the linear functional $u \mapsto p[u, v] \leq K \|u\|_X \|v\|_X$ and hence it is continuous, a similar argument shows $u \mapsto p[v, u]$ is also continuous. In particular $\lim_{n\to\infty} p[u_n, v] = p[u, v]$ and $\lim_{n\to\infty} p[v, u_n] = p[v, u]$.

Then $0 \leq q(u_n - u) = q(u_n) + q(u) - p[u, u_n] - p[u_n, u]$, so by taking the liminf on both sides of $p[u, u_n] + p[u_n, u] - q(u) \leq q(u_n)$ we find that $\liminf_n q(u_n) \geq q(u)$ □

Lemma 19. *Let q, p, with p symmetric, be as above, then $q \in C^1(X)$ and $D_u I(v) = 2p[u, v]$*

Proof. It holds that

$$q(u + tv) = q(u) + t^2 q(v) + 2tp[u, v]$$

$\square$

We also note the following

Fact 20. *Suppose that u minimizes the problem $\min_{u\in H} B[u,u] = \min_{u\in H} Q(u)$. Then u also minimzes the problem*

$$\min_{u\in H_0^1, u\not\equiv 0} \frac{B[u,u]}{\|u\|_2^2}$$

which can be seen immediately since B is bilinear and $\|au\|_2^2 = a^2 \|u\|_2^2$, so it scales correctly

We thus have that

Theorem 21. *Given the problem $\lambda_1 = \inf_{u\in H} Q(u)$ it holds that there is a minimizer w and this is a weak solution of*

$$\begin{cases} Lw = \lambda_1 w & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

Proof. We have that Q is weakly lower-semicontinuous by lemma 18,

$$\lim_{|u|_{H_0^1}\to\infty} Q(u) \geq C \lim_{|u|_{H_0^1}\to\infty} |u|_{H_0^1}^2 = \infty$$

and H is weakly closed by our above discussion. Thus by theorem 6 there is a minimizer w. We note that $\|u\|_2^2$ is the quadratic form induced by the bilinear form $\langle u, v\rangle_2$.

By fact 20 we see that w is a critical point of $\frac{Q(u)}{\|u\|_2^2}$ on the domain $u \in H_0^1, u \not\equiv 0$. Along with this the domain $u \in H_0^1, u \not\equiv 0$ is open, so our calculations for the claim at the end of section 3 should hold.

In particular for all $v \in H_0^1$ for small enough t $w + tv \not\equiv 0$, and so the quotient rule implies that

$$0 = \frac{B[w,v] \|w\|_2^2 - \langle w, v\rangle_2 Q(w)}{\|w\|_2^4}$$

but as $\|w\|_2 = 1$ this reduces to

$$0 = B[w, v] - \langle w, v \rangle_2 Q(w) = B[w, v] - \lambda_1 \langle w, v \rangle_2$$

In particular for all $v \in H_0^1$

$$B[w, v] = \lambda_1 \langle w, v \rangle_2$$

$\square$

Let us continue this analysis. Suppose we already have the eigenvalues $\lambda_1, ..., \lambda_n$ and corresponding eigenfunctions $e_1, ..., e_n$. Let $H = \{u \in H_0^1 : \|u\|_2 = 1 \text{ and } u \perp span\{e_1, ..., e_n\}\}$.

Lemma 22. *The set H is weakly closed.*

Proof. Clearly all that needs to be verified is that if $w_m \rightharpoonup w$ weakly and $w_m \perp span\{e_1, ..., e_n\}$ then $w \perp span\{e_1, ..., e_n\}$, since the other properties pass through weak limits.

To see this we note that for each $i = 1, ..., n$ $u \mapsto \langle u, e_i \rangle$ is a bounded linear functional. Hence $0 = \lim_{m\to\infty} \langle w_m, e_i \rangle = \langle w, e_i \rangle$ $\square$

Lemma 23. *It holds that $\lambda_{n+1} = \inf_{u \in H} Q(u)$.*

Proof. Note that $f \in H$ implies $f = \sum_{i=n+1}^{\infty} \langle f, e_i \rangle_2 e_i$ and that $\{e_i\}_{i=n+1}^{\infty}$ is an orthonormal basis of $\{u \in L^2 : u \perp span\{e_1, ..., e_n\}\}$. From here the analysis is the same as for theorem 17 $\square$

We have once more the fact:

Fact 24. *Suppose that u minimizes the problem $\min_{u \in H} B[u, u] = \min_{u \in H} Q(u)$. Then u also minimzes the problem*

$$\min_{u \in H_0^1, u \not\equiv 0, u \perp span\{e_1, ..., e_n\}} \frac{B[u, u]}{\|u\|_2^2}$$

and since $span\{e_1, ..., e_n\}$ is finite dimensional:

Fact 25. *The set $span\{e_1, ..., e_n\}$ is closed in H_0^1*

Hence we have that

Theorem 26. *It holds that the problem* $\lambda_{n+1} = \inf_{u\in H} Q(u)$ *has a minimizer* w *and this is a weak solution of*

$$\begin{cases} Lw = \lambda_{n+1} w & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

Proof. Take the derivative again using fact 24, which we can do since $u \in H_0^1, u \not\equiv 0, u \perp span\{e_1, ..., e_n\}$ is the intersection of two open sets, by fact 25. Lemma 23 implies that the number a such that

$$0 = B[w,v] - \langle w, v\rangle_2 Q(w) = B[w,v] - a\langle w, v\rangle_2$$

is actually $a = \lambda_{n+1}$ □

Hence we may deduce theorem 13 entirely using weak convergence methods and the calculus of variations.

As an aside it follows since $|\int_U u| \leq \int_U |u| \leq vol(U)\|u\|_2$ and $u \mapsto \int_U u$ is linear that $\dot{H}$ as in the Neumann problem, is weakly closed.

Exercise. This, along with the fact that $\dot{H}$ is an open subspace, may provide some inspiration to how to proceed in that case.

7 Continuous Variations of Eigenvalues

We now want to know how the Eigenvalues vary when the domain or the coefficients are changed.

Recall the general topological theorem:

Theorem 27. *Let* $f : X \times Y \to \mathbb{R}$ *be continuous in* x*, then* $g(x) = \inf_{y\in Y} f(x,y)$ *is upper-semicontinuous*

Now, if we suppose that $t \mapsto a_{ij}(*, t)$ and $t \mapsto c(*, t)$ are continuous in the C^2 topology, that is the topology generated by the norm

$$\|u(x)\|_{2,\infty} = \sup_{|\alpha|\leq 2, x\in \bar{U}} |u(x)|$$

then we want to show that the eigenvalues are continuous. Unfortunately, the lower-semicontinuity is quite difficult, but the upper-semicontinuity is much easier.

Theorem 28. *Let a_{ij}, c satisfy the continuity assumtions, $c \geq 0$, and suppose further that the uniform ellipticity assumtption holds for $t = t_0$. Then for small enough $|t_0 - t|$ and for $i > 0$ the function $t \mapsto \lambda_1(t)$ is upper-semicontinuous.*

Proof. It suffices to show that for each $u \in H = \{u \in H_0^1 : \|u\|_2 = 1\}$ the function $t \mapsto Q_t(u)$ is continuous near t_0 and the first eigenvalue exists for all such t, since then $\lambda_1(t) = \inf_{u \in H} Q_t(u)$ is upper-semicontinuous by theorem 27.
We have that for any symmetric matrix A it holds that $x^T A x \leq n^2 a|x|^2$ where $a = \sup_{ij} A_{ij}$. Put $M(t) = n^2 \sup_{x \in \bar{U}}^{i,j} (|a_{ij}(x,t)| + |c(x,t)|)$, so that it varies continuously and

$$C_{1,s}(t) = \sup_{x \in \bar{U}} |c(x,t) - c(x,s)|, C_{2,s}(t) = \sup_{x \in \bar{U}}^{i,j} |a_{ij}(x,t) - a_{ij}(x,s)|$$

so it holds that $\lim_{t \to s} C_{1,s}(t) = \lim_{t \to s} C_{2,s}(t) = 0$ and $|a_{ij}(x,t_0) - a_{ij}(x,t)|\xi_i\xi_j \leq n^2 C_{2,s}(t)|\xi|^2$ and so

$$a_{ij}(x,t)\xi_i\xi_j = a_{ij}(x,t_0)\xi_i\xi_j - (a_{ij}(x,t_0)\xi_i\xi_j - a_{ij}(x,t)\xi_i\xi_j)$$

$$\geq \theta|\xi|^2 - |a_{ij}(x,t_0) - a_{ij}(x,t)|\xi_i\xi_j \geq (\theta - n^2 C_{2,t_0}(t))|\xi|^2$$

$$B_t[u,v] \leq M(t) \|u\|_{H_0^1} \|v\|_{H_0^1}$$

$$Q_t(u) \geq (\theta - n^2 C_{2,t_0}(t)) \|u\|_{H_0^1}^2$$

hence for small enough $|t - t_0|$ it holds that B_t satisfies Lax-Milgram, so we may conclude that the first eigenvalue exists and is equal to the minimizer as in the last section. On the other hand for all s, t

$$|Q_t(u) - Q_s(u)| = \int (a_{ij}(x,t) - a_{ij}(x,s))u_i u_j + (c(x,t) - c(x,s))u \leq (C_{1,s}(t) + C_{2,s}(t)) \|u\|_{H_0^1}^2$$

and so we may conclude by theorem 27. □

From the paper [3], in particular lemma 3 we can assert the upper-semicontinuity of the higher eigenvalues as well. The reason we cannot with the current tools is that *span*$\{e_1, ..., e_n\}$, and in particular the set Y in theorem 27, varies with t

as well. Lemma 3 in the paper gives a characterization independent of t. A similar result, that unfortunately ruins the use of theorem 27 can be found in chapter 6 problem 13 in [1].

On the other hand the paper proves full continuity in a much different way than the techniques I have used, though theorem 28 in these notes leads quite naturally into nonlinear problems such as the Yamabe invariant, which is why I chose the C^2 topology.

Of course, I have only considered symmetric operators and so I would like to mention that [1] chapter 6 section 5 theorem 3 yields an analogue of the last part of theorem 21, while problem 14 in that same chapter yields an analogue of theorem 17

Finally, the dependence of the eigenvalues on the domain can be found in chapter 6 problem 15 in [1].

References

[1] Lawrence Craig Evans. *Partial differential equations.* American Mathematical Society, 2010.

[2] Gerald B. Folland. *Introduction to partial differential equations.* Prentice Hall of India, 2003.

[3] Jeong Hyeong Park. Continuous variation of eigenvalues and gÃ€rding's inequality. *Differential Geometry and its Applications*, 10(2):187â189, 1999.

[4] John Roe. *Elliptic operators, topology, and asymptotic methods.* Chapman amp; Hall/CRC, 2017.